SADLER UNIT 4 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 11: Simple harmonic motion

Exercise 11A

Question 1

$v = 6t\sqrt{16+t^2}$

a $a = 6t(16+t^2)^{-\frac{1}{2}} \times 2t + 6\sqrt{16+t^2} = \dfrac{12t^2}{\sqrt{16+t^2}} + 6\sqrt{16+t^2}$

When $t = 0$, $a = 6\sqrt{16} = 24\,\text{m/s}^2$

b Let $u = 16+t^2$, $\dfrac{du}{dt} = 2t$

$x = \int 6t\sqrt{16+t^2}\,dt = \int u^{\frac{1}{2}} 6t \times \dfrac{1}{2t}\,dt = 3\int u^{\frac{1}{2}} dt$

$= 3 \times \dfrac{2}{3} u^{\frac{3}{2}} + c = 2(16+t^2)^{\frac{3}{2}} + c$

When $t = 0$, $x = 8$.

$8 = 2(16^{\frac{3}{2}}) + c \quad \Rightarrow \quad c = -120$

$x = 2(16+t^2)^{\frac{3}{2}} - 120$

When $t = 3$, $x = 130\,\text{m}$.

Question 2

$$a = \frac{6t(t+1)^2}{5}$$

Let $u = t+1$, $\frac{du}{dt} = 1$

$$v = \int \frac{6t(t+1)^2}{5}dt = \int \frac{6(u-1)u^2}{5}dt = \frac{6}{5}\int (u^3 - u^2)dt$$
$$= \frac{6}{5}\left(\frac{u^4}{4} - \frac{u^3}{3}\right) + c = \frac{6}{5}\left(\frac{(t+1)^4}{4} - \frac{(t+1)^3}{3}\right) + c$$

When $t = 1$, $v = 2\,\text{m/s}$.

$$2 = \frac{6}{5}\left(\frac{2^4}{4} - \frac{2^3}{3}\right) + c$$
$$c = \frac{2}{5}$$

$$v = \frac{6}{5}\left(\frac{(t+1)^4}{4} - \frac{(t+1)^3}{3}\right) + \frac{2}{5}$$

When $t = 0$, $v = \frac{6}{5}\left(\frac{1^4}{4} - \frac{1^3}{3}\right) + \frac{2}{5} = 0.3\,\text{m/s}$

Question 3

$$x = 5 + 2\cos t$$
$$\frac{dx}{dt} = -2\sin t = v$$
$$\frac{dv}{dt} = -2\cos t = a$$

a When $t = \frac{\pi}{6}$, $v = -2\sin\frac{\pi}{6} = -1\,\text{m/s}$

b When $t = \frac{\pi}{2}$, $a = 0\,\text{m/s}^2$

Question 4

$v = 4\sin 2t$

$\dfrac{dv}{dt} = 8\cos 2t = a$

a When $t = \dfrac{\pi}{6}$, $a = 8\cos\dfrac{\pi}{3} = 4\ \text{m/s}^2$

b $\int 4\sin 2t\, dt = -2\cos 2t + c$

When $t = 0,\ x = 3.$

$x = -2\cos 2t + 5$

When $t = \dfrac{\pi}{2}$, $x = -2\cos\pi + 5 = 7\text{m}$.

Question 5

$a = 4\sin t\cos t$

$v = \int 4\sin t\cos t\, dt = 2\int \sin 2t\, dt = -\cos 2t + c$

When $t = 0$, $v = 3\text{m/s}$ so $c = 4$.

$v = -\cos 2t + 4$

$x = \int(-\cos 2t + 4)dt = -\dfrac{\sin 2t}{2} + 4t + c$

When $t = 0$, $x = 5\text{m}$

$x = -\dfrac{\sin 2t}{2} + 4t + 5$

a When $t = \dfrac{\pi}{3}$, $v = -\cos\left(\dfrac{2\pi}{3}\right) + 4 = 4\dfrac{1}{2}\text{m/s}$

b When $t = \dfrac{\pi}{3}$, $x = \left(5 + \dfrac{4\pi}{3} - \dfrac{\sqrt{3}}{4}\right)\text{m}$

Question 6

$v = 5 + x^2$

$\dfrac{dv}{dx} = 2x$

$a = \dfrac{dv}{dt}$

$\dfrac{dv}{dt} = \dfrac{dv}{dx} \times \dfrac{dx}{dt} = 2x(5 + x^2) = 10x + 2x^3$

When $x = 1,\ a = 12\,\text{m/s}$

Question 7

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Thus $v\dfrac{dv}{dx} = 3x^2 + 1$

$$\int v\ dv = \int (3x^2 + 1)dx$$

$$\frac{1}{2}v^2 = x^3 + x + c$$

When $x = 0,\ v = 2$

$$v^2 = 2x^3 + 2x + 4$$

When $x = 3,\ v = \sqrt{64} = 8\,\text{m/sec}.$

Question 8

$$a = \frac{dv}{dt} = v^2$$

$$\int \frac{1}{v^2}dv = \int dt$$

$$-v^{-1} = t + c$$

When $t = 2,\ v = 0.1$

$$-\frac{1}{0.1} = 2 + c$$

$$c = -12$$

$$-\frac{1}{v} = t - 12$$

a When $t = 10$

$$-\frac{1}{v} = 10 - 12$$

$$v = 0.5$$

b

$$v\frac{dv}{dx} = v^2$$

$$\int \frac{1}{v}dv = \int dx$$

$$\ln v = x + c$$

When $v = 0.1,\ x = 0$

$$\ln 0.1 = c$$

$$v = e^{(x + \ln 0.1)}$$

When $x = 2$

$$v = 0.1e^2$$

Question 9

$x = \dfrac{t+1}{2t+3}$

a $v = \dfrac{(2t+3)-2(t+1)}{(2t+3)^2} = \dfrac{1}{(2t+3)^2}$ m/s

$a = \dfrac{-4}{(2t+3)^3}$ m/s^2

b When $t=1$:

$x = \dfrac{2}{5} = 0.4$ m

$v = \dfrac{1}{25} = 0.04$ m/s

$a = -\dfrac{4}{125} = -0.032$m/s^2

Question 10

$h = 42 + 29t - 5t^2$

$0 = 42 + 29t - 5t^2$

$0 = (7-t)(6+5t)$

Hits the ground when $t = 7$ seconds.

$\dfrac{dh}{dt} = 29 - 10t$

When $t = 7$, the speed of the object is 41m/s.

Question 11

$x = t(16-t) \quad \Rightarrow \quad \dfrac{dx}{dt} = 16 - 2t$

a When $t = 20$, the speed of the particle is $|16 - 2 \times 20| = 24$ m/s

b The particle comes to rest when $16 - 2t = 0$

$t = 8$ m/s

When $t = 8,\ x = 8(16-8) = 64$ m

c When $t = 1,\ x = 1(16-1) = 15$ m

When $t = 5,\ x = 5(16-5) = 55$ m

The particle has travelled 55 – 15 = 40 m.

d When $t = 5,\ x = 5(16-5) = 55$ m

When $t = 8,\ x = 8(16-8) = 64$ m

When $t = 10,\ x = 10(16-10) = 60$ m

The particle has travelled 64 – 55 + 4 = 13 m.

Question 12

$$\int (6t-24)dt = 3t^2 - 24t + c$$

$$35 = 3(0)^2 - 24(0) + c$$

$$c = 35$$

$v = 3t^2 - 24t + 35$

$x = t^3 - 12t^2 + 35t + c$

When $t = 0,\ x = 0$ so $c = 0$.

$x = t^3 - 12t^2 + 35t = t(t-7)(t-5)$

The body is next at O when $t = 5$.

$v = 3(5)^2 - 24(5) + 35 = -10\,\text{m/s}$

Question 13

$v = 2\sin 2t$

a Velocity will be at a maximum when $\sin 2t$ is at its maximum value of 1.

$$\sin 2t = 1$$

$$2t = \frac{\pi}{2}$$

$$t = \frac{\pi}{4}$$

$$v = 2\sin 2\left(\frac{\pi}{4}\right) = 2\,\text{m/s}$$

b $a = \dfrac{dv}{dt} = (4\cos 2t)\,\text{m/s}^2$

c The acceleration will be at a maximum when $\cos 2t$ is at its maximum value of 1.

$\cos 2t = 1 \quad \Rightarrow \quad t = 0$

$a = 4\cos 2(0) = 4\,\text{m/s}^2$

d $x = \int 2\sin 2t\,dt = -\cos 2t + c$

The particle goes through the origin at time $t = 0$, so $c = 1$.

$x = (1 - \cos 2t)\,\text{m}$

e When $\cos 2t = -1$, the displacement will have its maximum value.

At $t = \dfrac{\pi}{2}$, $x = -\cos 2\left(\dfrac{\pi}{2}\right) + 1 = 2\,\text{m}$

Question 14

a $v = 3x + 2$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$
$$= (3x+2)(3) = (9x+6)\ \text{m/s}^2$$

b When $x = 4$, $v = 3(4) + 2 = 14$ m/s

When $x = 4$, $a = 9(4) + 6 = 42$ m/s^2

Question 15

Given $a = -(1+v^2)\text{m/s}^2$

$$a = -(1+v^2)$$
$$\frac{dv}{dt} = -(1+v^2)$$
$$\frac{dv}{dx}\frac{dx}{dt} = -(1+v^2)$$
$$\frac{dv}{dx}v = -(1+v^2)$$
$$\int \frac{v}{1+v^2}\,dv = \int (-1)dx$$
$$\frac{1}{2}\int \frac{2v}{1+v^2}\,dv = \int (-1)dx$$
$$\frac{1}{2}\ln\left|1+v^2\right| = -x + c$$
$$\ln\left|1+v^2\right| = -2x + c$$
$$e^{-2x+c} = 1+v^2$$
$$e^{-2x}e^c = 1+v^2$$
$$Ae^{-2x} = 1+v^2, \qquad \text{where } A = e^c$$

When $x = 0$, $v = 4$

$$Ae^0 = 1+4^2$$
$$A = 17$$
$$17e^{-2x} = 1+v^2$$

When $v = 1$

$$17e^{-2x} = 2$$

$$e^{-2x} = \frac{2}{17}$$

$$\ln e^{-2x} = \ln\frac{2}{17}$$

$$-2x = \ln\frac{2}{17}$$

$$x = -\frac{1}{2}\ln\frac{2}{17}$$

$$x = \frac{1}{2}\ln\left(\frac{2}{17}\right)^{-1}$$

$$x = \frac{1}{2}\ln\frac{17}{2}$$

When $v = 0,\ x = \frac{1}{2}\ln 17,$ which is a higher value than $\frac{1}{2}\ln\frac{17}{2},$

so there are no stationary points prior to $\frac{1}{2}\ln\frac{17}{2}.$

The exact distance the particle moves in reducing it's velocity to one quarter of its initial velocity is $0.5\ln(8.5)$.

Exercise 11B

Question 1

a $x = 5\sin 2t$

Amplitude = 5 m

Period = 2π ÷ 2 = π seconds

b $x = 4\sin 5t$

Amplitude = 4 m

Period = 2π ÷ 5 = 0.4π seconds

c $x = 2\cos 4t$

Amplitude = 2 m

Period = 2π ÷ 4 = 0.5π seconds

Question 2

a $\ddot{x} = -4x$

This is of the form $\ddot{x} = -k^2x,\ k = 2$

The period is $\dfrac{2\pi}{2} = \pi$ seconds

b $\ddot{x} = -x$

This is of the form $\ddot{x} = -k^2x,\ k = 1$

The period is $\dfrac{2\pi}{1} = 2\pi$ seconds

c $\ddot{x} = -25x$

This is of the form $\ddot{x} = -k^2x,\ k = 5$

The period is $\dfrac{2\pi}{5} = 0.4\pi$ seconds

Question 3

For simple harmonic motion, with the particle at O when timing commences, the equation of the displacement will take the form $x = a\sin kt$, where a is the amplitude and the period is equal to $2\pi \div k$.

a $\frac{2\pi}{k} = 4\pi,\ k = \frac{1}{2}$

$x = 1\ \sin\frac{t}{2} = \sin 0.5t$

b $\frac{2\pi}{k} = 4\pi,\ k = \frac{1}{2}$

$x = -1\ \sin\frac{t}{2} = -\sin 0.5t$

c $\frac{2\pi}{k} = \pi,\ k = 2$

$x = 3\sin 2t$

d $\frac{2\pi}{k} = 2,\ k = \pi$

$x = -0.5\sin \pi t$

Question 4

For simple harmonic motion, with the particle at its maximum distance from O when timing commences, the equation of the displacement will take the form $x = a\cos kt$, where a is the amplitude and the period is equal to $2\pi \div k$.

a $\frac{2\pi}{k} = \pi,\ k = 2$

$x = 2\cos 2t$

b $\frac{2\pi}{k} = 0.5\pi,\ k = 4$

$x = 1.5\cos 4t$

c $\frac{2\pi}{k} = 0.5,\ k = 4\pi$

$x = 0.5\cos 4\pi t$

Question 5

a $\frac{2\pi}{k} = \pi,\ k = 2$

$x = 2.5\sin 2t$

OR

$x = -2.5\sin 2t$

b $\dot{x} = 5\cos 2t$

When $t = \frac{\pi}{6}$

$\dot{x} = 5\cos 2\left(\frac{\pi}{6}\right) = 2.5\text{m/s}$

The speed of the body when $t = \frac{\pi}{6}$ is 2.5 m/s.

Question 6

a Amplitude $=\sqrt{5^2+3^2}=\sqrt{34}$ m

Period $=\dfrac{2\pi}{5}=0.4\pi$ seconds.

b Amplitude $=\sqrt{3^2+7^2}=\sqrt{58}$ m

Period $=\dfrac{2\pi}{2}=\pi$ seconds.

Question 7

a $x=4\sin\dfrac{\pi t}{10}$

To prove SHM we must show that $\ddot{x}=-k^2x$

We are given that $x=4\sin\dfrac{\pi t}{10}$.

Therefore $\dot{x}=\dfrac{4\pi}{10}\cos\dfrac{\pi t}{10}=\dfrac{2\pi}{5}\cos\dfrac{\pi t}{10}$

and $\ddot{x}=-\dfrac{2\pi^2}{50}\sin\dfrac{\pi t}{10}=-\dfrac{\pi^2}{25}\sin\dfrac{\pi t}{10}=-\dfrac{\pi^2}{100}\times 4\sin\dfrac{\pi t}{10}$

$$=-\left(\frac{\pi}{10}\right)^2\times 4\sin\frac{\pi t}{10}=-\left(\frac{\pi}{10}\right)^2 x$$

This is of the form $\ddot{x}=-k^2x$.

The motion is simple harmonic.

b Period $=2\pi\div\dfrac{\pi}{10}=20$ seconds

Amplitude = 4 m.

c When $t=0$, $x=4\sin(0)=0\,\text{m}$.

When $t=2$, $x=4\sin\left(\dfrac{2\pi}{10}\right)\approx 2.35\text{m}$

Question 8

a $x = 2\sin\frac{\pi t}{3}$

To prove SHM we must show that $\ddot{x} = -k^2 x$.

We are given that $x = 2\sin\frac{\pi t}{3}$.

Therefore, $\dot{x} = \frac{2\pi}{3}\cos\frac{\pi t}{3}$

and $\ddot{x} = -\frac{2\pi^2}{9}\sin\frac{\pi t}{3} = -\frac{\pi^2}{9}2\sin\frac{\pi t}{3}$

$$= -\left(\frac{\pi}{3}\right)^2 \times 2\sin\frac{\pi t}{3} = -\left(\frac{\pi}{3}\right)^2 x$$

This is of the form $\ddot{x} = -k^2 x$.

The motion is simple harmonic.

b Period $= 2\pi \div \frac{\pi}{3} = 6\,\text{seconds}$.

Amplitude = 2 m.

c In the first two seconds the body does not move in the same direction for the entire two seconds. To consider how far the body has moved, we need to consider the displacement at different times, when the body changes direction.

When $t = 0$, $x = 2\sin(0) = 0\,\text{m}$

When $t = 1.5$, $x = 2\sin\frac{3\pi}{6} = 2\sin\frac{\pi}{2} = 2\,\text{m}$.

When $t = 2$, $x = 2\sin\left(\frac{2\pi}{3}\right) = \sqrt{3}$

The distance travelled from $t = 1.5$ to $t = 2$, is $2 - \sqrt{3}$.

In the first two seconds the body moves $2 + 2 - \sqrt{3} = (4 - \sqrt{3})\,\text{m}$.

Question 9

a $x = 3\sin\left(2t + \frac{\pi}{6}\right)$

To prove SHM we must show that $\ddot{x} = -k^2 x$.

We are given that $x = 3\sin\left(2t + \frac{\pi}{6}\right)$

Therefore $\dot{x} = 6\cos\left(2t + \frac{\pi}{6}\right)$

and $\ddot{x} = -12\cos\left(2t + \frac{\pi}{6}\right) = -(2^2)3\sin\left(2t + \frac{\pi}{6}\right) = -(2)^2 x$

This is of the form $\ddot{x} = -k^2 x$.

The motion is simple harmonic.

b Period $= 2\pi \div 2 = \pi$ seconds.

Amplitude = 3 m.

c The body reaches maximum displacement when $\sin\left(2t + \frac{\pi}{6}\right) = 1$

In the first second this occurs at $t = \frac{\pi}{6}$

When $t = 0$, $x = 3\sin\left(2(0) + \frac{\pi}{6}\right) = 1.5\,\text{m}$

When $t = \frac{\pi}{6}$, $x = 3\sin\left(\frac{2\pi}{6} + \frac{\pi}{6}\right) = 3$

When $t = 1$, $x = 3\sin\left(2 + \frac{\pi}{6}\right) \approx 1.74\,\text{m}$

The distance travelled from $t = 0$ to $t = \frac{\pi}{6}$, is $3 - 1.5 = 1.5\,\text{m}$.

The distance travelled from $t = \frac{\pi}{6}$ to $t = 1$, is $3 - 1.74 \approx 1.26\,\text{m}$.

In the first second the body moves $1.5 + 1.26 = 2.76\,\text{m}$.

Question 10

a $x = \pm a\sin(kt + \alpha)$

Amplitude $= 4$, so $a = 4$

Period $= 2 = \frac{2\pi}{k}$, so $k = \pi$

$x = \pm 4\sin(\pi t + \alpha)$

When $t = 0,\ x = 2$

$4\sin\alpha = 2$

$\sin\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$ (velocity is positive), $\frac{5\pi}{6}$ (velocity is negative)

$x = 4\sin\left(\pi t + \frac{5\pi}{6}\right)$

b $x = 4\sin\left(\pi t + \frac{5\pi}{6}\right)$

$v = 4\pi\cos\left(\pi t + \frac{5\pi}{6}\right)$

When $t = \frac{1}{6}$

$v = 4\pi\cos\left(\frac{\pi}{6} + \frac{5\pi}{6}\right) = 4\pi\cos\pi = -4\pi$

The speed of the body when $t = \frac{1}{6}$ is 4π m/s.

Question 11

a $x = \pm a\sin(kt + \alpha)$

Amplitude $= 2$, so $a = 2$

Period $= \dfrac{2\pi}{5} = \dfrac{2\pi}{k}$, so $k = 5$

$x = \pm 2\sin(5t + \alpha)$

When $t = 0,\ x = \sqrt{2}$

$2\sin\alpha = \sqrt{2}$

$\sin\alpha = \dfrac{\sqrt{2}}{2}$

$\alpha = \dfrac{\pi}{4}$ (velocity is positive)

$x = 2\sin\left(5t + \dfrac{\pi}{4}\right)$

b $v = 10\cos\left(5t + \dfrac{\pi}{4}\right)$

This velocity (and hence the speed) will be at a maximum when $\cos\left(5t + \dfrac{\pi}{4}\right)$ is equal to 1.

The greatest speed attained by the body is 10 m/s.

c $a = -50\sin\left(5t + \dfrac{\pi}{4}\right)$

The greatest acceleration of the body will be at a maximum when $\sin\left(5t + \dfrac{\pi}{4}\right)$ is equal to –1.

The greatest acceleration of the body is 50 m/s^2.

Question 12

$\ddot{x} = -4x$

$x = a \sin kt$

$k = 2$

Amplitude = 0.6, so $a = 0.6$

$x = 0.6 \sin 2t$

a When $t = \frac{\pi}{6}$, $x = 0.6 \sin \frac{2\pi}{6} = \frac{6}{10} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{10}$ m

b When $t = \frac{\pi}{3}$, $x = 0.6 \sin \frac{2\pi}{3} = \frac{6}{10} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{10}$ m

c The distance of the body is 0.3 m from O when $0.3 = 0.6 \sin 2t$.

$$\sin 2t = \frac{1}{2}$$
$$2t = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$t = \frac{\pi}{12}, \frac{5\pi}{12}$$

The distance is also 0.3 m from O when $-0.3 = 0.6 \sin 2t$.

$$\sin 2t = -\frac{1}{2}$$
$$2t = \frac{7\pi}{6}$$
$$t = \frac{7\pi}{12}$$

i The body is 0.3 m from O for the first time after $\frac{\pi}{12}$ seconds.

ii The body is 0.3 m from O for the second time after $\frac{5\pi}{12}$ seconds.

iii The body is 0.3 m from O for the third time after $\frac{7\pi}{12}$ seconds.

Question 13

$\ddot{x} = -\pi^2 x$

$x = a\sin kt$

$k = \pi$

$a = 3$

$x = -3\sin \pi t$, (the function is negative as at $t = 0$, the velocity is negative).

a When $t = \frac{1}{3}$, $x = -3\sin\frac{\pi}{3} = -\frac{3\sqrt{3}}{2}$ m

b $x = -3\sin \pi t$

$v = -3\pi\cos \pi t$

When $t = \frac{1}{3}$

$v = -3\pi\cos\frac{\pi}{3} = -\frac{3\pi}{2}$ m/s

c When $t = \frac{1}{3}$, the speed of the body is $\frac{3\pi}{2}$ m/s

d $|v| = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad v = \pm\frac{\sqrt{3}}{2}$

$$-3\pi\cos \pi t = \pm\frac{\sqrt{3}}{2}$$

$$\cos \pi t = \pm\frac{1}{2}$$

$$\pi t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

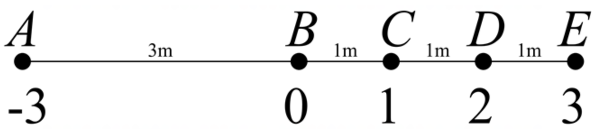
$$t = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}$$

The next time the body has a speed of $\frac{3\pi}{2}$ is after $\frac{2}{3}$ seconds.

Question 14

Suppose that the particle is at point *A* when $t = 0$.

The mean position is at *B*.



$$x = -3\cos kt$$

Period = π

$$\frac{2\pi}{k} = \pi$$

$$k = 2$$

$$x = -3\cos 2t$$

a The particle arrives at point *C* when $x = 1$.

$$-3\cos 2t = 1$$

$$\cos 2t = -\frac{1}{3}$$

$$2t \approx 1.91$$

$$t \approx 0.96 \text{ seconds}$$

It takes approximately 0.96 seconds to travel from *A* to *C*.

b The particle arrives at point *D* when $x = 2$.

$$-3\cos 2t = 2$$

$$\cos 2t = -\frac{2}{3}$$

$$2t \approx 2.30$$

$$t \approx 1.15 \text{ seconds}$$

$$1.15 - 0.96 = 0.19 \text{ seconds}$$

It takes approximately 0.19 seconds to travel from *C* to *D*.

c The particle arrives at point *D* when $x = 3$.

$$-3\cos 2t = 3$$

$$\cos 2t = -\frac{3}{3}$$

$$t \approx 1.57 \text{ seconds}$$

$$1.57 - 1.15 = 0.42 \text{ seconds}$$

It takes approximately 0.42 seconds to travel from *D* to *E*.

d The particle could travel from *D* to *E* and then back to *D*, this would take $2 \times 0.42 = 0.84$ seconds.

Alternatively, the particle could travel from *D* to *A* and then back to *D*, this would take $2 \times (\pi - 0.42)$ or approximately 5.44 seconds.

Question 15

$$x = 2\sin 4t$$

$$2\sin 4t = \pm 1.5$$

$$\sin 4t = \pm\frac{3}{4}$$

$$4t \approx 0.848, 2.294, 3.990, 5.435$$

$$t \approx 0.212, 0.573, 0.997, 1.359$$

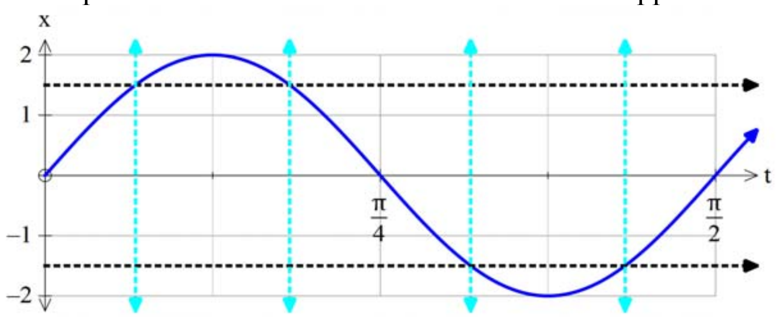
The particle is at least 1.5 m from O for

$0.573 - 0.212 + 1.359 - 0.997 = 0.72$ seconds.

(see the diagram below for the reasoning behind this calcualtion,

points on graph where $y \geq 1.5$ and the points where $y \leq -1.5$).

So the particle would be at least 1.5 m from O for approximately 0.72 seconds.



Question 16

$\ddot{x} = -4x$

Given $\ddot{x} = -k^2 x$ for a body in simple harmonic motion, $k = 2$.

$x = a\sin(kt + \alpha) = a\sin(2t + \alpha)$

$\dot{x} = v = 2a\cos(2t + \alpha)$

a When $t = 0$, $x = 0$.

$x = a\sin(2t + \alpha)$

$0 = a\sin\alpha \quad \Rightarrow \quad \alpha = 0$

$x = a\sin 2t$

When $t = 0$, $v = 4$

$v = 2a\cos(2t + \alpha)$

$4 = 2a\cos(0) \quad \Rightarrow \quad a = 2$

$x = 2\sin 2t$

b When $t = 0$, $v = 0$

$v = 2a\cos(2t + \alpha)$

$0 = 2a\cos(0 + \alpha)$

$\alpha = \frac{\pi}{2}$

When $t = 0$, $x = 4$.

$x = a\sin(2t + \alpha)$

$4 = a\sin\frac{\pi}{2} \quad \Rightarrow \quad a = 4$

$x = 4\sin\left(2t + \frac{\pi}{2}\right) = 4\cos 2t$

Question 17

$\ddot{x} = -64x$

For a mass in simple harmonic motion $\ddot{x} = -k^2 x$.

$k = 8$

a As the mass is pulled down 2 cm below equilibrium and released from rest the amplitude of the motion is 2 cm.

b $k = 8$

The period of the motion is $\frac{2\pi}{8} = \frac{\pi}{4}$ seconds.

c Reaches equilibrium quarter of the way through a full cycle, so at

$t = \frac{1}{4} \times \frac{\pi}{4} = \frac{\pi}{16}$ seconds.

d $x = -2\cos 8t$

$v = 16\sin 8t$

The mass will be at its maximum speed as it passes through the equilibrium level, the maximum speed is 16 cm/s.

e Maximum speed will occur when $\sin 8t$ is at its maximum value of 1, the maximum speed would then be 16 and half the maximum speed 8cm/s.

$8 = 16\sin 8t$

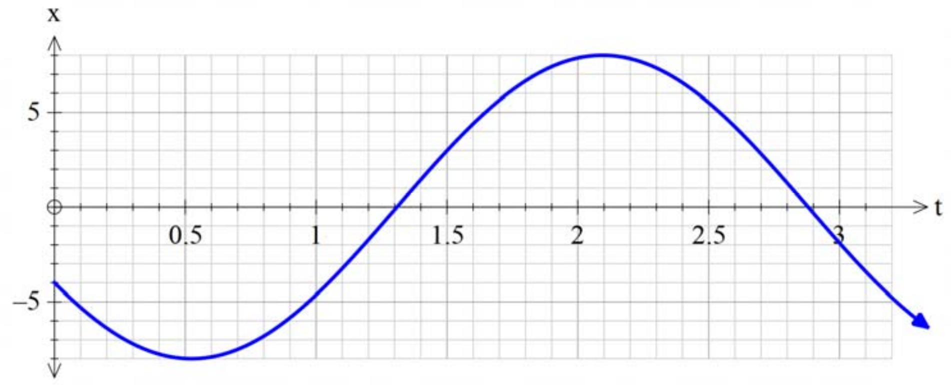
$\sin 8t = \frac{1}{2}$

$8t = \frac{\pi}{6}$

$t = \frac{\pi}{48}$ seconds.

Question 18

$x = -4\sqrt{3}\sin 2t - 4\cos 2t$



a When $t = 0$, $x = -4$.

The object is 4 m from O when $t = 0$.

b $x = -4\sqrt{3}\sin 2t - 4\cos 2t$

$$\dot{x} = -8\sqrt{3}\cos 2t + 8\sin 2t$$
$$\ddot{x} = 16\sqrt{3}\sin 2t + 16\cos 2t$$
$$= -4(-4\sqrt{3}\sin 2t - 4\cos 2t)$$
$$= -4x$$
$$\ddot{x} = -k^2 x$$
$$k = 2$$

c When $t = 0$, $x = -4$

When $t = 1.5$, $x = 2.982$

Observing the graph of the function shows that the function reaches its largest minimum value of –8 before reaching the value of 2.982, after 1.5 seconds.

This means that the object moves 4 + 8 + 2.982 metres, so approximately 14.98 m, in the first 1.5 seconds.

Question 19

$x = 3 + 4\sin \pi t$

a $x - 3 = 4\sin \pi t$

$p = 4\sin \pi t$

$\dot{p} = 4\pi \cos \pi t$

$\ddot{p} = -4\pi^2 \sin \pi t = -(\pi)^2 4\sin \pi t$

which satisfies $\ddot{p} = -k^2 p, k = \pi$.

b Period $= \dfrac{2\pi}{\pi} = 2$ seconds.

Amplitude = 4 m.

c We know that $x = 3 + 4\sin \pi t$, the mean value of the sine function is 0, so the mean position of the function in simple harmonic motion is 3 + 0, so 3 m.

d The maximum value of the function occurs when $\sin \pi t = 1$, $x = 3 + 4(1)$.

So 7 m is the greatest distance that the particle is from O.

Question 20

$x = 5 - 3\cos 2t$

a Let $s = x - 5$

$s = -3\cos 2t$

$\dot{s} = 6\sin 2t$

$\ddot{s} = 12\cos 2t = -2^2(-3\cos 2t)$

which satisfies $\ddot{s} = -k^2 s,\ k = 2$.

b The period is $\dfrac{2\pi}{2} = \pi$ seconds.

The amplitude is 3 m.

c We know that $x = 5 - 3\cos 2t$, the mean value of the cosine function is 0, so the mean position of the function in simple harmonic motion is 5 – 0, so 5 m.

d The least distance will occur when $\cos 2t$ has a value of 1, so 5 – 3(1) = 2 m.

Question 21

$v = \frac{1}{4}\cos t$

a $x = \frac{1}{4}\sin t + c$

When $t = 0$, $x = c$

When $t = 1$, $x \approx 0.21 + c$

Distance travelled is $0.21 + c - c = 0.21\,\text{m}$.

b The object reaches its maximum distance from O at $t = \frac{\pi}{2} \approx 1.57$.

When $t = 1.57$, $x = 0.25 + c$

When $t = 2$, $x = 0.227 + c$

Distance travelled in first two seconds is (0.25 – 0.23) + 0.25, so 0.27 m.

Question 22

$x = a\sin(kt + \alpha)$

$v = ak\cos(kt + \alpha)$

$20 = a\sin(kt + \alpha) \quad \Rightarrow \quad \frac{20}{a} = \sin(kt + \alpha)$

$30 = ak\cos(kt + \alpha) \quad \Rightarrow \quad \frac{30}{ak} = \cos(kt + \alpha)$

$\sin^2(kt + \alpha) + \cos^2(kt + \alpha) = \left(\frac{20}{a}\right)^2 + \left(\frac{30}{ak}\right)^2 = 1$

$24 = a\sin(kt + \alpha) \quad \Rightarrow \quad \frac{24}{a} = \sin(kt + \alpha)$

$14 = ak\cos(kt + \alpha) \quad \Rightarrow \quad \frac{14}{ak} = \cos(kt + \alpha)$

$\sin^2(kt + \alpha) + \cos^2(kt + \alpha) = \left(\frac{24}{a}\right)^2 + \left(\frac{14}{ak}\right)^2 = 1$

Solving gives

$a = 25,\ k = 2; \qquad a = 25,\ k = -2$

$a = -25,\ k = 2; \qquad a = -25,\ k = -2$

$\text{Period} = \frac{2\pi}{2} = \pi\ \text{seconds.}$

Amplitude is 25 m.

Question 23

$x = a\sin(kt+\alpha)$

$v = ak\cos(kt+\alpha)$

$0.6 = a\sin(kt+\alpha) \quad \Rightarrow \quad \dfrac{0.6}{a} = \sin(kt+\alpha)$

$0.75 = ak\cos(kt+\alpha) \quad \Rightarrow \quad \dfrac{0.75}{ak} = \cos(kt+\alpha)$

$\sin^2(kt+\alpha)+\cos^2(kt+\alpha) = \left(\dfrac{0.6}{a}\right)^2 + \left(\dfrac{0.75}{ak}\right)^2 = 1$

$0.39 = a\sin(kt+\alpha) \quad \Rightarrow \quad \dfrac{0.39}{a} = \sin(kt+\alpha)$

$1.56 = ak\cos(kt+\alpha) \quad \Rightarrow \quad \dfrac{1.56}{ak} = \cos(kt+\alpha)$

$\sin^2(kt+\alpha)+\cos^2(kt+\alpha) = \left(\dfrac{0.39}{a}\right)^2 + \left(\dfrac{1.56}{ak}\right)^2 = 1$

Solving gives

$a = 0.65,\ k = 3; \qquad a = 0.65,\ k = -3$

$a = -0.65,\ k = 3; \qquad a = -0.65,\ k = -3$

$\text{Period} = \dfrac{2\pi}{3}$ seconds.

Amplitude is 0.65 m.

Question 1

a $$\frac{d}{dx}(\ln y) = \frac{d}{dx}(3x^2)$$
$$\frac{1}{y}\frac{dy}{dx} = 6x$$
$$\frac{dy}{dx} = 6xy$$

b $$\frac{d}{dx}(4xy + y^5 - 15x) = \frac{d}{dx}(4\sin 2x)$$
$$4x\frac{dy}{dx} + 4y + 5y^4\frac{dy}{dx} - 15 = 8\cos 2x$$
$$\frac{dy}{dx}(4x + 5y^4) = 8\cos 2x - 4y + 15$$
$$\frac{dy}{dx} = \frac{15 - 4y + 8\cos 2x}{4x + 5y^4}$$

Question 2

$$\frac{dA}{dt} = -0.02A$$
$$\int \frac{1}{A}\,dA = \int -0.02\,dt$$
$$\ln A = -0.02t + c$$

$$A = e^{-0.02t+c} = e^c e^{-0.02t} = A_0 e^{-0.02t}$$
$$\frac{1}{2} = 1e^{-0.02t}$$
$$t = 34.66$$

So the half-life of the radioactive element is approximately 34.7 years.

Question 3

$v = 3\sin^2 t$

a The minimum value of the velocity will occur when $\sin t$ is zero, so the minimum velocity of the particle during motion is 0 m/s.

b $a = 6\sin t\cos t = 3(2\sin t\cos t) = (3\sin 2t)\text{m/s}^2$

c Acceleration is at its maximum value when $\sin t = \cos t$.

$\tan t = 1$

$t = \dfrac{\pi}{4}$

d $x = \int 3\sin^2 t \; dt = \dfrac{3}{2}\int (1-\cos 2t)dt = \dfrac{3}{2}\left(t - \dfrac{\sin 2t}{2}\right) + c$

When $t = 0,\ x = 0$. Hence $c = 0$.

$x = (1.5t - 0.75\sin 2t)$ m.

e When $t = \dfrac{\pi}{6}$,

$x = \dfrac{\pi}{4} - \dfrac{3}{4} \times \dfrac{\sqrt{3}}{2} \;\; = \dfrac{2\pi - 3\sqrt{3}}{8}$ m

Question 4

$v = 3x^2 - 2$

a $a = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx}$

$= (3x^2 - 2)(6x)\text{m/s}^2$

b When $x = 1$, $v = 1$m/s and $a = 6\,\text{m/s}^2$.

Question 5

$$\frac{dy}{dx} = \frac{2x}{3y^2}$$

$$\int 3y^2 dy = \int 2x\,dx$$

$$y^3 = x^2 + c$$

The point (2, 1) lies on the curve.

$$1^3 = 2^2 + c$$

$$c = -3$$

$$y^3 = x^2 - 3$$

Question 6

$$\frac{dh}{dt} = 1.2\,\text{m/sec}$$

$$h = 50\sin\theta$$

$$\frac{dh}{d\theta} = 50\cos\theta$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt} = \frac{1}{50\cos\theta} \times 1.2 = \frac{3}{125\cos\theta}$$

When $h = 40$, $\theta = 0.9273$

$$\frac{d\theta}{dt} = \frac{3}{125\cos(0.9273)} = 0.04\,\text{rad/sec}$$

$$\overline{AB} = \sqrt{50^2 - h^2}$$

$$\frac{d\overline{AB}}{dh} = \frac{1}{2}(50^2 - h^2)^{-\frac{1}{2}}(-2h) = \frac{-h}{\sqrt{50^2 - h^2}}$$

When $h = 40$ m

$$\frac{d\overline{AB}}{dh} = \frac{-40}{\sqrt{50^2 - 40^2}} = -\frac{40}{30} = -1\frac{1}{3}$$

$$\frac{d\overline{AB}}{dt} = \frac{d\overline{AB}}{dh} \times \frac{dh}{dt} = -\frac{4}{3} \times 1.2 = -1.6\,\text{m/s}$$

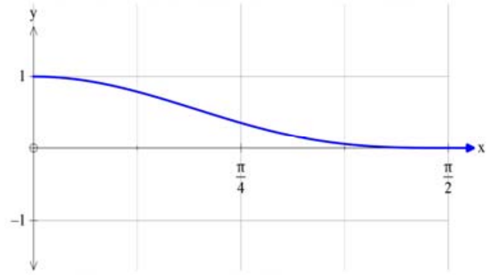
$\overline{AB}$ is decreasing in length at a rate of 1.6 m/s as the height of the kite increases.

B approaches *A* at a rate of 1.6 metres per second.

Question 7

$$\int_3^5 \pi y^2 dx = \int_3^5 \pi\left(4\sqrt{x}\right)^2 dx = \pi\int_3^5 16x\,dx = \left[8\pi x^2\right]_3^5 = 128\pi\,\text{units}^3$$

Question 8



$$\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos x \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)\,dx$$

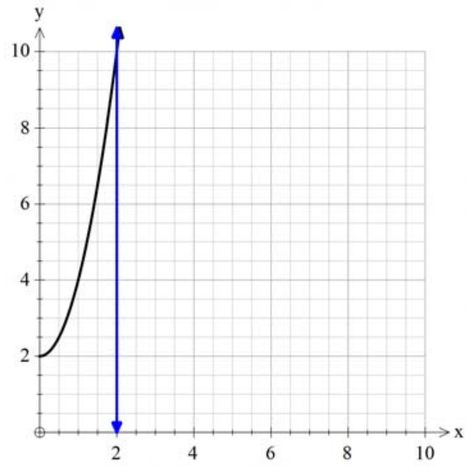
Let $u = \sin x \qquad \frac{du}{dx} = \cos x$

$$\int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)\,dx = \int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)\frac{dx}{du}\,du$$

$$= \int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)\frac{1}{\cos x}\,du$$

$$= \int_0^{1} (1-u^2)du = \left[u - \frac{u^3}{3}\right]_0^1 = \frac{2}{3}\text{units}^2$$

Question 9



a $$\int_0^2 2(x^2+1)\text{d}x = \left[\frac{2x^3}{3} + 2x\right]_0^2 = \frac{16}{3} + 4 = \frac{28}{3}\text{units}^2$$

b $$\int_0^2 \pi y^2 dx = \pi\int_0^2 4(x^2+1)^2 dx = 4\pi\int_0^2 (x^4+2x^2+1)dx$$

$$= 4\pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^2 = 4\pi\left(\frac{32}{5} + \frac{16}{3} + 2 - 0\right)$$

$$= \frac{824\pi}{15}\text{units}^3$$

c $$y = 2(x^2+1)$$

$$\frac{y}{2} = x^2 + 1$$

$$x^2 = \frac{y}{2} - 1$$

$$\int_2^{10} \pi x^2 dy = \pi\int_2^{10}\left(\frac{y}{2} - 1\right)dy = \pi\left[\frac{y^2}{4} - y\right]_2^{10}$$

$$= \pi\left(25 - 10 - (1-2)\right)$$

$$= 16\pi\,\text{units}^3$$

$$\pi\int_0^{10} x^2 dy = \pi\int_0^{10} 4\,dy = \pi\left[4y\right]_0^{10} = 40\pi$$

$$40\pi - 16\pi = 24\pi$$

The volume of the solid of revolution formed by rotation the region described in part **a** through 360° about the y-axis.

Question 10

$a = (6t + 4)\ \text{m/s}^2$

$v = \int (6t+4)dt = 3t^2 + 4t + c$

$x = \int (3t^2 + 4t + c)dt = t^3 + 2t^2 + ct + k$

The particle travels 32 m in the third second, so it follows that:

$(3)^3 + 2(3)^2 + (3)c + k - \left[2^3 + 2(2^2) + 2c + k\right] = 32$

$$c + 29 = 32$$
$$c = 3$$
$$v = 3t^2 + 4t + 3$$

When $t = 1$

$$v = 10\text{m/s}$$

Question 11

a $x = 2\sin 4t$

$v = 8\cos 4t$

$a = -32\sin 4t = -16(2\sin 4t) = -(4^2)x = -k^2x$

$k = 4$

Period $= \dfrac{2\pi}{k} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$ seconds .

This shows that the object has simple harmonic motion with period = $\dfrac{\pi}{2}$ seconds.

When $t = 0$, $x = 0\,\text{m}$ and the distance from the mean position to O is 0 m.

b $x = 5\cos 3t$

$v = -15\sin 3t$

$a = -45\cos 3t = -9(5\cos 3t) = -k^2x$

$k = 3$

Period $= \dfrac{2\pi}{3}$ seconds

This shows that the object has simple harmonic motion with period $= \dfrac{2\pi}{3}$ seconds.

When $t = 0$, $x = 5\,\text{m}$ and the distance from the mean position to O is 0 m.

c $x = 2\cos 2t + 4\sin 2t$

$v = -4\sin 2t + 8\cos 2t$

$a = -8\cos 2t - 16\sin 2t = -4(2\cos 2t + 4\sin 2t)$

$= -2^2(2\cos 2t + 4\sin 2t) = -k^2 x$

$k = 2$

$\text{Period} = \dfrac{2\pi}{2} = \pi$ seconds.

This shows that the object has simple harmonic motion with period = π seconds.
When $t = 0$, $x = 2$ m and the distance from the mean position to O is 0 m.

$x = 1 + 3\sin 5t$

Let $s = x - 1$

$s = 3\sin 5t$

$\dot{s} = 15\cos 5t$

$\ddot{s} = -75\sin 5t = -5^2(3\sin 5t) = -k^2 x$

$k = 5$

Period $= \dfrac{2\pi}{5}$ seconds.

This shows that the object has simple harmonic motion with period $= \dfrac{2\pi}{5}$ seconds.

When $t = 0$, $x = 1$ m and the distance from the mean position to O is 1 m.

Question 12

Particle A

$x = c\sin k_1 t$

Looking at the graph for the displacement against velocity of Particle A, the amplitude of the displacement is 5, so $c = 5$.

$x = 5\sin k_1 t$

$v = 5k_1 \cos k_1 t$

From the graph, $5k_1 = 10$

$k_1 = 2$

Time period for Particle A is $\dfrac{2\pi}{2} = \pi$ seconds.

Particle B

$x = d\sin k_2 t$

$v = dk_2 \cos k_2 t$

$a = -dk_2^2 \sin k_2 t$

From the graph:

$dk_2 = 3$

$dk_2^2 = 1.5$

Solving gives:

$k_2 = \dfrac{1}{2}, \quad d = 6$

Time period for Particle B is $2\pi \div \dfrac{1}{2} = 4\pi$ seconds